Copernicus's epicycles from Newton's gravitational force law via linear perturbation theory in geometric algebra

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Abstract. We derive Copernicus's epicycles from Newton's gravitational force law by assuming that a planet's orbit is a perturbed circular orbit, with the perturbation defined to be co-rotating with the said orbit. We substitute this orbit expression into Newton's gravitation law and showed that the perturbation satisfies the linear part of Hill's oscillator equation for lunar motion. We solve this oscillator equation using an exponential Fourier series and impose the boundary conditions at the aphelion and perihelion to derive the Copernicus's formulas for the eccentric, deferent, and epicycle. We show that for small eccetricity, the Copernican orbit expression also leads to Kepler's law of areas for planetary motion. The formalism we use is the Clifford (geometric) algebra $\mathcal{C}l_{2,0}$.

1 Introduction

In many introductory physics courses, especially those dealing with the history and philosophy of science, the Copernican model is taught conceptually but not mathematically.[1] And in undergraduate and graduate physics courses, it is not even mentioned at all. One possible reason is that, unlike in the case of Kepler's ellipse, Copernicus' epicycles has not been rigorously derived before from first principles, i.e., from Newton's laws of motion and gravitation. So our aim in this paper is to present this derivation. But before we do so, let us first review the Copernican model.

Copernicus believed that planets orbit around the sun. If the orbit of a planet is circular with the sun at the center, then the planet's position in complex form is

$$\hat{r} = r_0 e^{\hat{\imath}\omega_0 t}, \tag{1}$$

where r_0 and ω_0 are the planet's orbital radius and frequency, respectively. But because planets sometimes move closest to the sun (perihelion) and sometimes farthest (aphelion), Copernicus displaced the center of the planet's circular orbit a little away from the sun to a new point called the eccentric, so that the new position $\hat{r}$ of the planet is

$$\hat{r} = r_{-1} + r_0 e^{\hat{\imath}\omega_0 t}, \tag{2}$$

where r_{-1} is the distance of the eccentric from the sun.

Actually, the eccentric hypothesis in Eq. (2) is not Copernicus' original idea but was already known more than a thousand years prior by Ptolemy (though he assumed that the earth is at rest and not the sun as in Copernicus). In fact, if we factor out the exponential $e^{\hat{\imath}\omega_0 t}$ in Eq. (2), we would arrive at Ptolemy's theorem applied by Copernicus in his heliocentric theory:

$$\hat{r} = r_{-1} + r_0 e^{\hat{\imath}\omega_0 t} = (r_0 + r_{-1}\, e^{-\hat{\imath}\omega_0 t}) e^{\hat{\imath}\omega_0 t}. \tag{3}$$

In Ptolemaic terms, $r_{-1}e^{-\hat{\imath}\omega_0 t}$ is called an epicycle and Eq. (3) is called the eccentric-epicycle equivalence theorem. (The actual theorem is stated geometrically.[2, 3]) Notice that the theorem essentially states the equivalence of the description of the planet's position in the inertial frame (left hand side) and in the rotating frame (quantity in parenthesis on the right hand side).

Yet Eq. (2) is still not consistent with the numerical data. Ptolemy resolved this problem by assuming that the planet's circular orbit is uniform not with respect to the orbit's geometric center but on another point called the equant[4, 5]. Though this construction saves the appearances, Copernicus claimed that the equant goes against the idea of uniform circular motion and for him this is "not sufficiently pleasing to the mind"[6]. To remedy this aesthetic difficulty, Copernicus added on top of his original circle in the inertial frame another circle with twice the frequency[7]. In complex notation, we write

$$\hat{r} = r_{-1} + r_0 e^{\hat{\imath}\omega_0 t} + r_1 e^{2\hat{\imath}\omega_0 t} \tag{4}$$

where $r_{-1} = -3r_1$.

In terms of the orbit's semimajor axis and eccentricity ϵ, the Copernican expression in Eq. (4) becomes

$$\hat{r} = A(\frac{3}{2}\epsilon + e^{\hat{\imath}\omega_0 t} - \frac{1}{2}\epsilon\, e^{2\hat{\imath}\omega_0 t}), \tag{5}$$

as given by Gallavotti[5]. Notice that Eq. (5) is different from that derived from Kepler's elliptical orbit for small eccentricity:[8]

$$\hat{r} = A(\epsilon(1-\hat{\imath}) + e^{\hat{\imath}\omega_0 t} + \epsilon(1+\hat{\imath})e^{2\hat{\imath}\omega_0 t}). \tag{6}$$

In this paper, our aim is to show that the Copernican expression in Eq. (5) is a consequence of Newton's gravitational force law.

We shall divide the paper into four sections. The first section is Introduction. In the second section, we shall present a brief tutorial on the Clifford (geometric) algebra $\mathcal{C}l_{2,0}$ for the plane[9, 10, 11, 12, 13], which combines scalars, vectors and imaginary numbers. We shall show how the exponential Fourier series are related to eccentrics, deferents, and epicycles.[14, 15] In the third section, we shall introduce a pertubation in the planet's position in the frame co-rotating with the planet's unperturbed circular orbit and substitute the result to the vector form of Newton's law of gravitation. We shall show that the perturbation in complex form satisfies the linear harmonic oscillator equation

$$0 = \ddot{\hat{s}} + 2\hat{\imath}\omega_0\dot{\hat{s}} - \frac{3}{2}\omega_0^2(\hat{s} + \hat{s}^*), \tag{7}$$

whose scalar and imaginary parts are

$$\begin{aligned} 0 &= \ddot{x}_s - 2\omega_0\dot{y}_s - 3\omega_0^2 x_s, && (8)\\ 0 &= \ddot{y}_s + 2\omega_0\dot{x}_s. && (9) \end{aligned}$$

These equations are the linear part of Hill's equations for lunar motion[16, 17]. We shall solve Eq. (7) using exponential Fourier series and impose the boundary conditions at the aphelion and perihelion to derive the Fourier coefficients of the Copernican orbit. And the fourth section is Conclusions.

2 Geometric Algebra

2.1 Vectors and Complex Numbers

The Clifford (geometric) algebra $\mathcal{C}l_{2,0}$ is an associative algebra generated by two vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ that correspond to the basis vectors along the $x-$ and $y-$axis in the Cartesian coordinate system. The vectors satisfy the orthonormality relation

$$\mathbf{e}_\mu\mathbf{e}_\nu + \mathbf{e}_\nu\mathbf{e}_\mu = 2\delta_{\mu\nu}, \tag{10}$$

for $\mu, \nu = 1, 2$. That is,

$$\begin{aligned} \mathbf{e}_1^2 = \mathbf{e}_2^2 = 1, && (11)\\ \mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_2\mathbf{e}_1. && (12) \end{aligned}$$

The first equation algebraically defines $\mathbf{e}_1$ and $\mathbf{e}_2$ as unit vectors by setting their squares to unity; the second equation defines the vectors as mutually orthogonal by making their product anticommute.

Let us define the unit bivector

$$\hat{\imath} = \mathbf{e}_1\mathbf{e}_2. \tag{13}$$

From the orthonormality axiom in Eq. (10), it is easy to see that $\hat{\imath}$ is an imaginary number,

$$\hat{\imath}^2 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_1(\mathbf{e}_2\mathbf{e}_2)\mathbf{e}_1 = -\mathbf{e}_1\mathbf{e}_1 = -1, \tag{14}$$

that anticommutes with vectors $\mathbf{e}_1$ and $\mathbf{e}_2$:

$$\begin{aligned} \mathbf{e}_1\hat{\imath} &= \mathbf{e}_2 = -\hat{\imath}\mathbf{e}_1, && (15)\\ \mathbf{e}_2\hat{\imath} &= -\mathbf{e}_1 = -\hat{\imath}\mathbf{e}_2. && (16) \end{aligned}$$

Notice that right-multiplying $\hat{\imath}$ to a vector rotates it counterclockwise by $\pi/2$.

In general, a vector $\mathbf{a}$ in the two-dimensional space spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$ is given by

$$\mathbf{a} = a_x\mathbf{e}_1 + a_y\mathbf{e}_2 = \mathbf{e}_1\hat{a} = \hat{a}^*\mathbf{e}_1, \tag{17}$$

where

$$\begin{aligned} \hat{a} &= a_x + a_y\hat{\imath}, && (18)\\ \hat{a}^* &= a_x - a_y\hat{\imath}. && (19) \end{aligned}$$

Equations (17) to (19) relates the vector $\mathbf{a}$ to the complex number $\hat{a}$ and its complex conjugate $\hat{a}^*$.

If vector $\mathbf{b} = b_x\mathbf{e}_1 + b_y\mathbf{e}_2$, then the product of vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b} = \hat{a}^*\hat{b}, \tag{20}$$

where

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= a_x b_x + a_y b_y, && (21)\\ \mathbf{a}\wedge\mathbf{b} &= (a_x b_y - a_y b_x)\hat{\imath} && (22) \end{aligned}$$

are the scalar (dot) and imaginary (bivector or planar) parts of the product $\mathbf{ab} = \hat{a}^*\hat{b}$. Notice that the magnitude of the wedge product is that of the cross product $\mathbf{a}\times\mathbf{b}$. (Geometrically, we say that $\mathbf{a}\times\mathbf{b}$ is the vector perpendicular to the oriented plane $\mathbf{a}\wedge\mathbf{b}$, though technically, $\mathbf{a}\times\mathbf{b}$ is not defined in $\mathcal{C}l_{2,0}$—only in $\mathcal{C}l_{3,0}$).

2.2 Circles, Epicycles, and Fourier Series

Because $\hat{\imath}$ is an imaginary number, then Euler's theorem holds:

$$e^{\hat{\imath}\theta} = \cos\theta + \hat{\imath}\sin\theta, \tag{23}$$

where θ is a real number. If we left-multiply Eq. (23) by $\mathbf{e}_1$, we get

$$\mathbf{e}_1 e^{\hat{\imath}\theta} = \mathbf{e}_1\cos\theta + \mathbf{e}_2\sin\theta, \tag{24}$$

where we used Eq. (15). Equation (24) states that $\mathbf{e}_1 e^{\hat{\imath}\theta}$ is the vector $\mathbf{e}_1$ rotated counterclockwise by an angle θ (assuming that $\mathbf{e}_1$ points to the right and $\mathbf{e}_2$ points up).

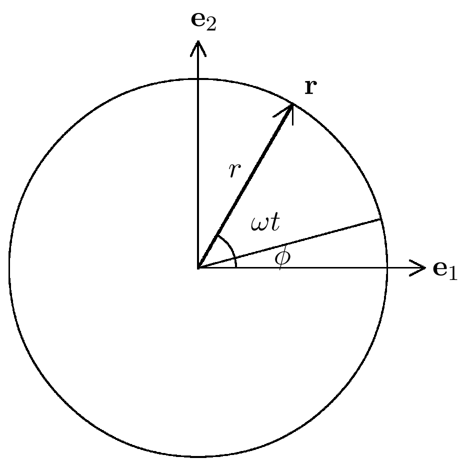

Fig. 1. The vector $\mathbf{r} = \mathbf{e}_1 r e^{i(\omega t+\phi)}$.

The theorem in Eq. (24) enables us to express the position $\mathbf{r}$ of a point in uniform circular motion as

$$\mathbf{r} = \mathbf{e}_1 r e^{\hat{\imath}(\omega t+\phi)} = \mathbf{e}_1 r\cos(\omega t+\phi) + \mathbf{e}_2 r\sin(\omega t+\phi), \tag{25}$$

where r is radius, ω is the angular frequency, and ϕ is the rotational phase angle. Another way to express $\mathbf{r}$ is

$$\mathbf{r} = \mathbf{e}_1\hat{r}\hat{\psi}, \tag{26}$$

where

$$\hat{r} = re^{\hat{\imath}\phi}, \tag{27}$$
$$\hat{\psi} = e^{\hat{\imath}\omega t} \tag{28}$$

are the complex radius and rotor (rotation operator), respectively. (See Fig. (1))

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be two rotating vectors:

$$\mathbf{r}_1 = \mathbf{e}_1\hat{r}_1\hat{\psi}_1 = \mathbf{e}_1 r_1 e^{i(\omega_1 t+\phi_1)}, \tag{29}$$
$$\mathbf{r}_2 = \mathbf{e}_1\hat{r}_2\hat{\psi}_2 = \mathbf{e}_1 r_2 e^{i(\omega_2 t+\phi_2)}. \tag{30}$$

If we displace their sum by a vector $\mathbf{r}_0$,

$$\mathbf{r}_0 = \mathbf{e}_1\hat{r}_0 = \mathbf{e}_1 r_0 e^{\hat{\imath}\phi_0}, \tag{31}$$

we arrive at

$$\mathbf{r} = \mathbf{e}_1(\hat{r}_0 + \hat{r}_1\hat{\psi}_1 + \hat{r}_2\hat{\psi}_2). \tag{32}$$

One way to simplify Eq. (32) is to set $\omega_1 = \omega$ and $\omega_2 = 2\omega$. So using the definition of the rotor ψ in Eq. (28), we get

$$\mathbf{r} = \mathbf{e}_1(\hat{r}_0 + \hat{r}_1\hat{\psi} + \hat{r}_2\psi^2). \tag{33}$$

The zeroth harmonic is the eccentric; the first, the deferent; and the second, the epicycle. In general, we may express the position $\mathbf{r}$ in time t as an infinite Fourier series:

$$\mathbf{r} = \mathbf{e}_1\sum_{k=-\infty}^{\infty}\hat{r}_k\hat{\psi}^k = \sum_{k=-\infty}^{\infty}\mathbf{e}_1 r e^{\hat{\imath}(k\omega t+\phi_k)}. \tag{34}$$

Equation (34) represents the Copernican ideal of decomposing an orbit as a sum of epicycles with harmonic frequencies.

3 Copernican Dynamics

3.1 Uniform Circular Orbit

In Newton's law of gravitation, the equation of motion of a planet of mass m revolving around the sun of mass M is

$$\ddot{\mathbf{r}} = -GM\frac{\mathbf{r}}{|\mathbf{r}|^3}, \tag{35}$$

where $\mathbf{r}$ is the position of the planet with respect to the the sun at the origin.

One to solution to Eq. (35) is a circular orbit:

$$\mathbf{r} = \mathbf{r}_0 = \mathbf{e}_1\hat{r}_0\hat{\psi}_0 = \mathbf{e}_1 r_0 e^{\hat{\imath}(\omega_0 t+\phi_0)}, \tag{36}$$

where r_0 is the orbital radius, ω_0 is the orbital frequency, and ϕ_0 is the orbital phase angle.

To verify this claim, we first take the derivatives in time of the position vector $\mathbf{r}$:

$$\dot{\mathbf{r}} = \omega_0\mathbf{e}_1\hat{\imath}\hat{r}_0\hat{\psi}_0 = -\hat{\imath}\omega_0\mathbf{r}_0, \tag{37}$$
$$\ddot{\mathbf{r}} = -\omega_0^2\mathbf{r}_0. \tag{38}$$

Next, we take the square of the position $\mathbf{r}$ by using the conjugation theorem in Eq. (17):

$$\mathbf{r}^2 = \mathbf{e}_1\hat{r}_0\hat{\psi}_0\mathbf{e}_1\hat{r}_0\hat{\psi}_0 = \hat{r}_0^*\hat{\psi}_0^{-1}\hat{r}_0\hat{\psi}_0 = \hat{r}_0^*\hat{r}_0 = r_0^2, \tag{39}$$

so that $|\mathbf{r}| = r_0$ as we expect. And finally, we substitute Eqs. (37) to (39) back to Eq. (35) to arrive at

$$\omega_0^2 = \frac{GM}{r_0^3}, \tag{40}$$

which is the circular orbit condition.

3.2 Linear Perturbation Theory

Let us assume that the solution to Eq. (35) may be expressed as a sum of a circular orbital position $\mathbf{r}_0$ and its small correction $\mathbf{r}_1$:

$$\mathbf{r} = \mathbf{r}_0 + \lambda \mathbf{r}_1, \tag{41}$$

where λ is a perturbation parameter that will be set to unity later. If we also assume that the perturbation $\mathbf{r}_1$ lies in the same orbital plane as the original circular orbit $\mathbf{r}_0$ in Eq. (36) and co-rotating with it, then we may write $\mathbf{r}_1$ as

$$\mathbf{r}_1 = \mathbf{e}_1 \hat{s} \hat{\psi}_0, \tag{42}$$

where $\hat{s}$ is a complex function. Hence,

$$\mathbf{r} = \mathbf{e}_1 (\hat{r}_0 + \lambda \hat{s}) \hat{\psi}_0. \tag{43}$$

Taking the first and second time derivatives of the position $\mathbf{r}$ in Eq. (43), we get

$$\begin{aligned} \dot{\mathbf{r}} &= \mathbf{e}_1 (\lambda \dot{\hat{s}} + \hat{\imath}\omega_0 (\hat{r}_0 + \lambda \hat{s})) \hat{\psi}_0, & (44) \\ \ddot{\mathbf{r}} &= \mathbf{e}_1 (\lambda \ddot{\hat{s}} + 2\lambda \hat{\imath}\omega_0 \dot{\hat{s}} - \omega_0^2 (\hat{r}_0 + \lambda \hat{s})) \hat{\psi}_0. & (45) \end{aligned}$$

Equation (45) provides the expansion of the left side of Newton's gravitation law in Eq. (35).

On the other hand, to rewrite the right side of the gravitation law, we need first to take the square of the position vector $\mathbf{r}$ in Eq. (43) and retain only the terms up to first order in λ:

$$\mathbf{r}^2 = (\hat{r}_0 + \lambda \hat{s})^* (\hat{r}_0 + \lambda \hat{s}) \approx r_0^2 + \lambda (\hat{r}_0^* \hat{s} + \hat{r}_0 \hat{s}^*). \tag{46}$$

Raising both sides of Eq. (46) to $-3/2$ power and employing the binomial theorem, we get

$$\frac{1}{|\mathbf{r}|^3} \approx \frac{1}{r_0^3} \left(1 - \lambda \frac{3}{2r_0} (\hat{\eta}_0^* \hat{s} + \hat{\eta}_0 \hat{s}^*) \right), \tag{47}$$

where

$$\hat{\eta}_0 = e^{\hat{\imath}\phi_0}. \tag{48}$$

Multiplying Eq. (47) by the position $\mathbf{r}$ in Eq. (43) yields

$$\frac{\mathbf{r}}{|\mathbf{r}|^3} \approx \frac{1}{r_0^3} \mathbf{r}_0 + \mathbf{e}_1 \frac{\lambda}{r_0^3} \left(\hat{s} - \frac{3}{2r_0} (\hat{s} + \hat{\eta}_0^2 \hat{s}^*) \right) \hat{\psi}_0, \tag{49}$$

where we retained only the terms up to first order in λ.

Now, substituting Eqs. (43) and (47) back to the gravitation law in Eq. (35), we arrive at

$$0 = \ddot{\hat{s}} + 2\hat{\imath}\omega_0 \dot{\hat{s}} - \frac{3}{2} \omega_0^2 (\hat{s} + \hat{\eta}_0^2 \hat{s}^*). \tag{50}$$

If we set $\phi_0 = 0$ (this means that orbit is not tilted, as we shall show later), so that $\hat{\eta}_0 = e^{i\phi_0} = 1$, we get Hill's oscillator equation in Eq. (7).

4 Copernican Analysis

4.1 Epicyclical Fourier Series

We assume that the solution to the orbital harmonic oscillator equation in Eq. (50) is an exponential Fourier series with ω_0 as the fundamental angular frequency:

$$\hat{s} = \sum_{k=-\infty}^{\infty} \hat{a}_k \hat{\psi}_0^k. \tag{51}$$

The time derivatives of $\hat{s}$ are

$$\begin{aligned} \dot{\hat{s}} &= \hat{\imath}\omega_0 \sum_{k=-\infty}^{\infty} k \hat{a}_k \hat{\psi}_0^k, & (52) \\ \ddot{\hat{s}} &= -\omega_0^2 \sum_{k=-\infty}^{\infty} k^2 \hat{a}_k \hat{\psi}_0^k, & (53) \end{aligned}$$

while the conjugate of $\hat{s}$ is

$$\hat{s}^* = \sum_{k=-\infty}^{\infty} \hat{a}_k^* \hat{\psi}_0^{-k} = \sum_{k=-\infty}^{\infty} \hat{a}_{-k}^* \hat{\psi}_0^k. \tag{54}$$

Substituting Eqs. (51) to (54) back to Eq. (50), we get

$$0 = \sum_{k=-\infty}^{\infty} ((k^2 + 2k + \frac{3}{2}) \hat{a}_k + \frac{3}{2} \hat{\eta}_0^2 \hat{a}_k^*) \hat{\psi}_0^k, \tag{55}$$

after factoring out $-\omega_0^2$ and rearranging the terms. Because the rotors $\hat{\psi}_0^k$ are orthonormal in the Fourier sense, then Eq. (55) holds only if the coefficient of $\hat{\psi}_0^k$ is zero for all k:

$$0 = (k^2 + 2k + \frac{3}{2}) \hat{a}_k + \frac{3}{2} \hat{\eta}_0^2 \hat{a}_k^*. \tag{56}$$

Solving for the coefficient $\hat{a}_k$ in Eq. (56), we get

$$\hat{a}_{-k} = -\frac{2}{3} \hat{\eta}_0^2 (k^2 + 2k + \frac{3}{2}) \hat{a}_k^*. \tag{57}$$

Replacing the index k by $-k$,

$$a_k = -\frac{2}{3} \hat{\eta}_0^2 (k^2 - 2k + \frac{3}{2}) \hat{a}_{-k}^*, \tag{58}$$

and substituting the result back in Eq. (57), we arrive at

$$0 = (k^2 + 2k + \frac{3}{2})(k^2 - 2k + \frac{3}{2}) - \frac{9}{4} = k^2 (k^2 - 1), \tag{59}$$

after factoring out $\hat{a}_k$ and rearranging the terms. Hence,

$$k = \{-1, 0, 1\}. \tag{60}$$

Because the values of the index k are limited by Eq. (60), then the Fourier series for the perturbation $\hat{s}$ in Eq. (51) simplifies to

$$\hat{s} = \hat{a}_{-1}\hat{\psi}_0^{-1} + \hat{a}_0 + \hat{a}_1\hat{\psi}_0. \tag{61}$$

The relationship between the coefficients $\hat{a}_{-1}$ and $\hat{a}_1$ in Eq. (61) may be obtained by setting $k = 1$ in Eq. (57):

$$\hat{a}_{-1} = -3\hat{\eta}_0^2\hat{a}_1^*. \tag{62}$$

Similarly, the condition for $\hat{a}_0$ is

$$\hat{a}_0 = -\hat{\eta}_0^2\hat{a}_0^*. \tag{63}$$

This is satisfied in three possible ways:

$$\hat{a}_0 = \{\pm\hat{\imath}\hat{\eta}_0, 0\}. \tag{64}$$

Substituting the expression for $\hat{s}$ in Eq. (61) back to the position vector expression in Eq. (43), we get

$$\mathbf{r} = \mathbf{e}_1(\hat{a}_{-1} + (\hat{r}_0 + \hat{a}_0)\hat{\psi}_0 + \hat{a}_1\hat{\psi}_0^2). \tag{65}$$

Let us count the number of unknowns in this equation. The coefficient $\hat{a}_{-1}$ is related to $\hat{a}_1$ by Eq. (62). The angular frequency ω_0 in $\hat{\psi}_0 = e^{i\omega_0 t}$ is related to the radius r_0 of $\hat{r}_0 = r_0\hat{\eta}_0 = r_0 e^{i\phi_0}$ by Eq. (40). The phase angle ϕ_0 of $\hat{r}_0$ is related to that of $\hat{a}_0$ by Eq. (64). Thus, there are five unknowns in Eq. (65): $a_{1x}, a_{1y}, r_0, \phi_0$, and a_0.

However, the orbit of a planet in the plane is completely specified in two ways: (a) given the position $\mathbf{r}_1$ and the velocity $\mathbf{v}_1$ at a particular time t_1 or (b) given the positions $\mathbf{r}_1$ and $\mathbf{r}_2$ at their respective times t_1 and t_2. In other words, there are two constraint vector equations that are equivalent to four scalar equations for the components. These four equations can only determine four unknowns and not five, so one of our unknowns is superfluous and this must be a_0 because $\hat{a}_0 = 0$ is a possibility in Eq. (64). Thus, Eq. (65) reduces to

$$\mathbf{r} = \mathbf{e}_1(\hat{a}_{-1} + \hat{r}_0\hat{\psi}_0 + \hat{a}_1\hat{\psi}_0^2). \tag{66}$$

Because of the similarity of Eq. (66) to Eq. (33), we recognize $\mathbf{e}_1\hat{a}_{-1}$ as the eccentric, $\mathbf{e}_1\hat{r}_0\hat{\psi}_0$ as the deferent, and $\mathbf{e}_1\hat{a}_1\hat{\psi}_0^2$ as the epicycle in the Copernican model.

4.2 Boundary Conditions: Aphelion and Perihelion

Suppose that at $t = 0$, the planet is at its aphelion position $\mathbf{r}_a$ at a distance r_a from the sun at a counterclockwise angle γ from the positive $x-$axis; while at $t = \tau/2 = \pi/\omega_0$ the planet is at its perihelion position $\mathbf{r}_p$ at a distance r_p from the sun at a similar angle from the negative $x-$axis. Imposing these boundary conditions on the position vector $\mathbf{r}$ in Eq. (66) yields two simultaneous equations:

$$\begin{aligned}\mathbf{r}_a &= \mathbf{e}_1 r_a e^{\hat{\imath}\gamma} = \mathbf{e}_1(\hat{a}_{-1} + \hat{r}_0 + \hat{a}_1), & (67)\\ \mathbf{r}_p &= -\mathbf{e}_1 r_p e^{\hat{\imath}\gamma} = \mathbf{e}_1(\hat{a}_{-1} - \hat{r}_0 + \hat{a}_1). & (68)\end{aligned}$$

Factoring out $\mathbf{e}_1$ from Eqs. (67) and (68) and using the expression for $\hat{a}_{-1}$ in Eq. (62), we get

$$\begin{aligned}r_a e^{\hat{\imath}\gamma} &= -3\hat{\eta}_0^2\hat{a}_1^* + \hat{r}_0 + \hat{a}_1, & (69)\\ -r_p e^{\hat{\imath}\gamma} &= -3\hat{\eta}_0^2\hat{a}_1^* - \hat{r}_0 + \hat{a}_1, & (70)\end{aligned}$$

which are two simultaneous equations for $\hat{r}_0$ and $\hat{a}_1$.

Deferent. To solve for $\hat{r}_0$, we take the difference of Eqs. (69) and (70) to get

$$\hat{r}_0 = r_0\hat{\eta}_0 = r_0 e^{\hat{\imath}\phi_0} = \frac{1}{2}(r_a + r_p)e^{\hat{\imath}\gamma}, \tag{71}$$

so that

$$\begin{aligned}r_0 &= \frac{1}{2}(r_a + r_p), & (72)\\ \phi_0 &= \gamma. & (73)\end{aligned}$$

Thus, the radius r_0 of the deferent circle is the length of the semimajor axis of the orbit; the phase angle ϕ_0 is the angle of inclination of the semimajor axis from the $x-$axis along $\mathbf{e}_1$.

Epicycle. To solve for the coefficient $\hat{a}_1$, we add the Eqs. (67) and (68) to obtain

$$\frac{1}{2}(r_a - r_p)e^{\hat{\imath}\gamma} = -3\hat{\eta}_0^2\hat{a}_1^* + \hat{a}_1. \tag{74}$$

Because $\hat{a}_1 = a_x + \hat{\imath}a_y$ cannot be readily isolated, we separate the real and imaginary parts of Eq. (74) to get

$$\begin{aligned}\frac{1}{2}(r_a - r_p)\cos\gamma &= a_{1x}(-3\cos 2\gamma + 1)\\ &\quad + a_{1y}(-3\sin 2\gamma), & (75)\\ \frac{1}{2}(r_a - r_p)\sin\gamma &= a_{1x}(-3\sin 2\gamma)\\ &\quad + a_{1y}(3\cos 2\gamma + 1), & (76)\end{aligned}$$

where we used the relation $\phi_0 = \gamma$ in Eq. (73). Solving for the components a_{1x} and a_{1y} and combining the results, we arrive at

$$\hat{a}_1 = -\frac{1}{4}(r_a - r_p)e^{\hat{\imath}\gamma}. \tag{77}$$

Equation (77) states that the radius a_1 of the epicycle $\hat{a}_1$ is one-fourth the difference between the aphelion distance r_a and the perihelion distance r_p. Note the negative sign.

Eccentric. After knowing $\hat{a}_1$, we use the coefficient relation in Eq. (62) to solve for $\hat{a}_{-1}$:

$$\hat{a}_{-1} = \frac{3}{4}(r_a - r_p)e^{\hat{\imath}\gamma}. \tag{78}$$

Equation (78) states that the length a_{-1} of the eccentric is three-fourth the difference between the aphelion distance r_a and the perihelion distance r_p.

4.3 Copernican Orbit

We now substitute the expressions $\hat{a}-$coefficients in Eqs. (77) and (78) and that of $\hat{r}_0$ in Eq. (71) back to the expression for the position $\mathbf{r}$ in Eq. (66) to get

$$\mathbf{r} = \mathbf{e}_1 e^{\hat{\imath}\gamma}(\frac{3}{4}(r_a - r_p) + \frac{1}{2}(r_a + r_p)\hat{\psi}_0 - \frac{1}{4}(r_a - r_p)\hat{\psi}_0^2). \quad (79)$$

If the semimajor axis' inclination angle $\gamma = \phi_0 = 0$, then Eq. (79) reduces to

$$\mathbf{r} = \mathbf{e}_1(\frac{3}{4}(r_a - r_p) + \frac{1}{2}(r_a + r_p)\hat{\psi}_0 - \frac{1}{4}(r_a - r_p)\hat{\psi}_0^2). \quad (80)$$

Equation (80) is our desired approximation of a planet's orbit around the sun using eccentric, deferent, and epicycle in terms of the planet's aphelion r_a and perihelion r_p. (See Fig. (2))

If we employ the definitions of the Keplerian semimajor axis A and eccentricity ϵ,

$$\begin{aligned} A &= \frac{1}{2}(r_a + r_p), && (81)\\ \epsilon &= \frac{r_a - r_p}{r_a + r_p} && (82) \end{aligned}$$

then we may rewrite Eq. (80) as

$$\mathbf{r} = \mathbf{e}_1 A(\frac{3}{2}\epsilon + \hat{\psi}_0 - \frac{1}{2}\epsilon\hat{\psi}_0^2), \quad (83)$$

which is Eq. (5). Or in Cartesian coordinates,

$$\begin{aligned} x &= A(\frac{3}{2}\epsilon + \cos(\omega_0 t) - \frac{1}{2}\epsilon\cos(2\omega_0 t)), && (84)\\ y &= A(\sin(\omega_0 t) - \frac{1}{2}\epsilon\sin(2\omega_0 t)). && (85) \end{aligned}$$

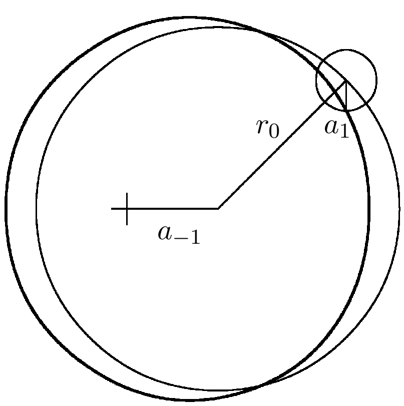

Fig. 2. Compass-and-straightedge plotting of a Copernican orbit with eccentric distance a_1, deferent radius r_0, and epicycle radius a_1. The orbit's eccentricity is $\epsilon = 1/3$.

The time derivative of the planet's position $\mathbf{r}$ in Eq. (83) is

$$\mathbf{v} = \mathbf{e}_1\hat{\imath}\omega A(\hat{\psi}_0 - \epsilon\hat{\psi}_0^2). \quad (86)$$

Left-multiplying this by $\mathbf{r}$,

$$\begin{aligned} \mathbf{r}\mathbf{v} &= \hat{\imath}\omega_0 A^2(\frac{3}{2}\epsilon + \hat{\psi}_0^{-1} - \frac{1}{2}\epsilon\hat{\psi}_0^{-2})(\hat{\psi}_0 - \epsilon\hat{\psi}_0^2)\\ &= \hat{\imath}\omega_0 A^2[-\frac{1}{2}\epsilon\hat{\psi}_0^{-1} + (1 + \frac{1}{2}\epsilon^2)\\ &\quad + \frac{1}{2}\epsilon\hat{\psi}_0 - \frac{3}{2}\epsilon^2\hat{\psi}_0^2], && (87) \end{aligned}$$

and separating the scalar and imaginary parts of the result, we arrive at

$$\begin{aligned} \mathbf{r}\cdot\mathbf{v} &= \omega_0 A^2(-\epsilon\sin(\omega_0 t) + \frac{3}{2}\epsilon^2\sin(2\omega_0 t)), && (88)\\ \mathbf{r}\wedge\mathbf{v} &= \hat{\imath}\omega_0 A^2((1 + \frac{1}{2}\epsilon^2) - \frac{3}{2}\epsilon^2\cos(2\omega_0 t)). && (89) \end{aligned}$$

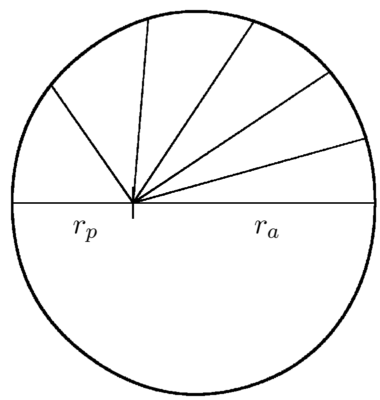

Fig. 3. The position of a planet in Copernican orbit from $t = 0$ to $t = \tau/2$ at a time interval of $\tau/12$. The orbit's eccentricity is $\epsilon = 1/3$.

For nearly circular orbits, the eccentricity $\epsilon \approx 0$ (ϵ is 0.0167 for earth and 0.0068 for Venus). So dropping the ϵ^2 terms in Eqs. (88) and (89), we arrive at

$$\begin{aligned} \mathbf{r}\cdot\mathbf{v} &= -\omega_0 A^2\epsilon\sin(\omega_0 t), && (90)\\ \mathbf{r}\wedge\mathbf{v} &= \hat{\imath}\omega_0 A^2. && (91) \end{aligned}$$

The first equation means that the position and velocity of a planet are perpendicular at the aphelion ($t = 0$) and perihelion ($t = \tau/2 = \pi/\omega_0$); the second equation implies that the oriented area

$$A = \mathbf{r}\wedge(\mathbf{v}\delta t) = \hat{\imath}\omega_0 A^2\delta t \quad (92)$$

swept by the radius vector $\mathbf{r}$ for a small interval of time δt is constant, which is Kepler's second law (we can always perform an integral to show the validity of the law for large time intervals). (See Fig. (3))

5 Summary and Conclusions

In this paper, we derived the Copernican system of epicycles from Newton's gravitational force law in vector form via linear perturbation theory in Clifford (geometric) algebra $\mathcal{C}l_{2,0}$ of the plane. We assumed that the planet's orbit is a perturbed circular orbit, where the perturbation is defined as a vector co-rotating with the original orbit. We substituted this expression into Newton's gravitation law. Using binomial expansion, we showed that this perturbation may be represented by a complex function $\hat{s}$ that satisfies the linearized form of Hill's equation for lunar motion. This equation is a linear harmonic oscillator with imaginary damping term and an extra forcing term that is proportional to the conjugate $\hat{s}^*$.

We solved this oscillator equation using exponential Fourier series with the frequency ω_0 of the unperturbed circular orbit as the fundamental frequency. We showed that only three harmonics are allowed: -1, 0, and 1. This result makes the planet's position as an exponential Fourier series with three harmonics: 0, 1, 2—corresponding to the planet's eccentric, deferent, and epicycle. We determined the values of the Fourier coefficients by imposing that the planet is at its aphelion at $t = 0$ and at its perihelion at $t = \tau/2 = \pi/\omega_0$. And from this we derived Gallavotti's expression for the Copernican orbit in terms of its semimajor axis A and eccentricity ϵ.

We also computed the dot and wedge products of the planet's position and velocity. We showed that for small eccentricity ϵ, the dot product is proportional to $-\sin(\omega_0 t)$; the wedge product is constant, $\hat{\imath}\omega_0 A^2$, which implies that the planet's position vector sweeps out equal areas in equal times, as given by Kepler's second law.

Acknowledgments

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